

Convex risk measures for good deal bounds

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Abstract

We study convex risk measures describing the upper and lower bounds of a good deal bound, which is a subinterval of a no-arbitrage pricing bound. We call such a convex risk measure a good deal valuation and give a set of equivalent conditions for its existence in terms of market. A good deal valuation is characterized by several equivalent properties and in particular, we see that a convex risk measure is a good deal valuation only if it is given as a risk indifference price. An application to shortfall risk measure is given. In addition, we show that the no-free-lunch (NFL) condition is equivalent to the existence of a relevant convex risk measure which is a good deal valuation. The relevance turns out to be a condition for a good deal valuation to be reasonable. Further we investigate conditions under which any good deal valuation is relevant.

Keywords: Convex risk measure, Good deal bound, Orlicz space, Risk indifference price, Fundamental theorem of asset pricing

1 Introduction

The no-arbitrage framework in mathematical finance is not sufficient for providing a unique price for a given contingent claim in an incomplete market. Instead provided is only a no-arbitrage pricing bound. Since it is in general too wide to be useful in financial practice, needed is an alternative way to find nice candidates of prices of contingent claims. As a method to give a sharper pricing bound, the framework of no-good-deal has been discussed in much literature; for example, [1] [5] [3] [6] [7] [8] [9] [11] [17]

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[19] [22] [25]. The no-arbitrage pricing bound for a claim is obtained by excluding prices which enable either a seller or buyer to enjoy an arbitrage opportunity by trading the claim and selecting a suitable portfolio strategy. The price in a market should be consistent with this bound to make the market viable. On the other hand, an upper (resp. a lower) good deal bound may be interpreted as determined by the seller's (resp. the buyer's) attitude to the risk associated with the claim. This can be considered as a generalization of the both pricing principle of no-arbitrage and exponential utility indifference valuation. Denote by $a(x)$ such an upper bound for a claim x . The functional a is supposed to have the following properties:

1. $a(0) = 0$,
2. $a(x) \leq a(y)$ if $x \leq y$,
3. $a(x + c) = a(x) + c$ for any $c \in \mathbb{R}$,
4. $a(\lambda x + (1 - \lambda)y) \leq \lambda a(x) + (1 - \lambda)a(y)$ for any $\lambda \in [0, 1]$

for any claims x and y . In the second property, the inequality $x \leq y$ is in the almost sure sense, where we regard the claims as random variables. In the third, the element $c \in \mathbb{R}$ stands for a deterministic cash-flow. The last one represents the risk-aversion of the seller taking into account the impact of diversification. In brief, we suppose that ρ_a defined as $\rho_a(x) := a(-x)$ is a normalized convex risk measure. If we impose additionally the positive homogeneity: $a(\lambda x) = \lambda a(x)$ for all x and $\lambda \geq 0$, which implies the subadditivity: $a(x + y) \leq a(x) + a(y)$ for all x and y , then ρ_a becomes a coherent risk measure. By the same sort argument as above, a functional b which refers to a lower good deal bound is given by a normalized convex risk measure ρ_b as $b(x) = -\rho_b(x)$.

A good deal bound should be a subinterval of the no-arbitrage pricing bound, so not every convex risk measure yields a good deal bound. The aim of this paper is to characterize such a convex risk measure, which we call a good deal valuation (GDV hereafter); we define GDV as a normalized convex risk measure ρ with the Fatou property such that for any claim x , the value $\rho(-x)$ lies in the no-arbitrage pricing bound of x . This definition of GDV is given from sellers' viewpoint; for a GDV ρ and a claim x , $a(x) := \rho(-x)$ serves as an ask price of x . Nevertheless, it is easy to see that if ρ is a GDV, then $b := -\rho$ gives bid prices. We impose the Fatou property as a natural continuity condition for good deal bounds.

First we investigate equivalent conditions for the existence of a GDV. Among others, we show that a GDV exists under a condition weaker than

the no-arbitrage one, which means that there may be GDVs even if the underlying market admits an arbitrage opportunity. Further we study equivalent conditions for a given ρ to be a GDV. In particular, we see that any GDV is given as a risk indifference price. The concept of risk indifference price has been undertaken by [26]. There is much literature on this topic ([14] [20] [23] among others). Some of the above papers observe that a risk indifference price provides a good deal bound. Our assertion is that its reverse implication also holds true, which seems a new insight.

As mentioned before, GDV may exist even in markets with free lunch. We observe the equivalence between the no-free-lunch condition (NFL) and the existence of a relevant GDV, that is a relevant convex risk measure which is a GDV. This could be considered as a version of Fundamental Theorem of Asset Pricing (FTAP). Moreover as a version of Extension Theorem, we see that the relevance of a GDV is equivalent to that the extended market by the GDV satisfies NFL. We see also that the relevance is equivalent to the no-near-arbitrage condition (NNA) introduced by [25]. We give an example (Example 4.9) which shows that NFL for the original market does not ensure NNA in general for a given GDV. We investigate conditions under which any GDV is relevant, and illustrate some examples related to this topic.

Now we mention the preceding results on FTAP from the viewpoint of good deal bound. Kreps [21] introduced NFL and proved FTAP as well as Extension Theorem. Černý and Hodges [8] established the framework of good deal bound and gave a version of Extension Theorem. Jaschke and Küchler [17] showed that good deal bounds are essentially equivalent to coherent risk measures and gave a variant of FTAP. Staum [25] extended their results to the noncoherent case. Bion-Nadal [5] introduced a dynamic version and gave an associated FTAP. In [17] and [25], an acceptance set reflecting an investor's preference is given first, and a convex risk measure induced by it is considered as a functional describing a good deal bound. Our approach is different, although we treat very similar problems. In our study, a convex risk measure is given first, and necessary and sufficient conditions for the given convex risk measure to be a GDV is discussed. This approach is in the same spirit as [5]. Our results provide a deeper understanding of a convex risk measure as a pricing functional in a market. Although our framework appears to be static, an extension to the dynamic framework of [5] can be done in a straightforward manner. A detailed comparison with [25] and [5] will be given in Remarks 3.5, 4.4 and 4.7.

In Section 2, we describe our model and prepare notation. In particular, we introduce the definitions and some basic properties of superhedging cost and risk indifference price. Main results are given in Sections 3 and 4.

2 Preliminaries

Here we introduce our framework and several basic results.

2.1 The Orlicz space

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. The Orlicz space L^Ψ with Young function Ψ is defined as the set of the random variables X such that there exists $c > 0$,

$$\mathbb{E}[\Psi(cX)] < \infty.$$

Here we call $\Psi : \mathbb{R} \rightarrow \mathbb{R} \cup \{\infty\}$ a Young function if it is an even convex function with $\Psi(0) = 0$, $\Psi(x) \uparrow \infty$ as $x \uparrow \infty$ and $\Psi(x) < \infty$ for x in a neighborhood of 0. It is a Banach lattice with the gauge norm

$$\|X\| := \inf\{c > 0; \mathbb{E}[\Psi(X/c)] \leq 1\}$$

and pointwise ordering in the almost sure sense. In the case of $\Psi = \Psi_\infty$:

$$\Psi_\infty(x) := \begin{cases} 0 & \text{if } |x| \leq 1, \\ \infty & \text{otherwise} \end{cases}$$

we have $L^\Psi = L^\infty$. Further, for $\Psi_p(x) := |x|^p$ with $p \geq 1$, we have $L^{\Psi_p} = L^p$. The Orlicz heart M^Ψ is a subspace of L^Ψ defined as

$$M^\Psi := \{X \in L^\Psi \mid \mathbb{E}[\Psi(cX)] < \infty \text{ for all } c > 0\}.$$

In this paper we consider the set of the future cash-flows L to be either L^Ψ or M^Ψ with a fixed Young function Ψ . This specification would be justified by noting that L becomes a linear space of random variables with natural ordering and sufficiently abstract in that it incorporates L^p spaces with $1 \leq p \leq \infty$. More importantly, a Young function Ψ may be connected to a utility function u as $\Psi(x) = -u(-|x|)$ and then L becomes a suitable space where expected utility maximization is considered (see e.g., [4]). Note that the case of exponential utility is covered. Our treatment and results do not depend on a specific choice of Ψ . This generality is indeed necessary to derive a conclusion which does not depend on a specific choice of utility function.

Let $M \subset L$ be the set of the 0-attainable claims. Each element of M represents a future payoff which investors can super-replicate with 0 initial endowment. Simultaneously, M might be regarded as the set of strategies

which investors can take. We suppose that M is a convex cone including L_- , where we denote L_+ (resp. L_-) := $\{x \in L | x \geq 0$ (resp. \leq) $\}$.

Let L_+^* be the set of all positive linear functionals on L . Remark that any element of L_+^* is continuous by the Namioka-Klee theorem (see [4] for an extended result). The both cases of $L = L^\Psi$ and $L = M^\Psi$ are treated in a unified way in the following. Let $L^\dagger := L^{\Psi^\dagger}$, where Ψ^\dagger is the complimentary function of Ψ defined as

$$\Psi^\dagger(y) := \sup_{x \in \mathbb{R}} \{xy - \Psi(x)\}.$$

Define a set of probability measures $\mathcal{P} := \{Q \ll \mathbb{P} | dQ/d\mathbb{P} \in L^\dagger\}$. Further, let $\bar{L}^* := \{g \in L_+^* | g(1) = 1, g(m) \leq 0 \text{ for any } m \in M\}$, $\mathcal{Q} := \{Q \in \mathcal{P} | dQ/d\mathbb{P} \in \bar{L}^*\}$, and $\mathcal{Q}^e := \{Q \in \mathcal{Q} | Q \sim \mathbb{P}\}$. For $Q \in \mathcal{P}$, denote by \mathbb{E}_Q the corresponding expectation operator. By Young's inequality:

$$\frac{xy}{ab} \leq \Psi\left(\frac{x}{a}\right) + \Psi^\dagger\left(\frac{y}{b}\right)$$

for any $x, y \in \mathbb{R}$ and $a, b > 0$, the operation \mathbb{E}_Q enables us to identify \mathcal{P} with a subset of L_+^* .

2.2 Convex risk measure

Here we collect several notions and results on convex risk measures which we utilize in this paper. A convex risk measure ρ is a $(-\infty, +\infty]$ -valued functional on L satisfying

properness: $\rho(0) < \infty$,

monotonicity: $\rho(x) \geq \rho(y)$ if $x \leq y$,

cash-invariance: $\rho(x + c) = \rho(x) - c$ for any $c \in \mathbb{R}$,

convexity: $\rho(\lambda x + (1 - \lambda)y) \leq \lambda\rho(x) + (1 - \lambda)\rho(y)$ for any $\lambda \in [0, 1]$,

for any $x, y \in L$. A convex risk measure ρ is a **coherent risk measure** if it satisfies in addition,

positive homogeneity: $\rho(cx) = c\rho(x)$ for any $x \in L$ and any $c > 0$.

Theorem 2.1 (Biagini and Frittelli [4]) *Let ρ be a convex risk measure. Then,*

$$\rho(-x) = \max_{g \in L_+^*, g(1)=1} \{g(x) - \rho^*(g)\}$$

for $x \in \text{Int}\{\rho < \infty\}$, where for $g \in L_+^*$,

$$\rho^*(g) := \sup_{x \in L} \{g(x) - \rho(-x)\}.$$

A convex risk measure ρ is said to have **the Fatou property** if for any increasing sequence $\{x_n\} \subset L$ with $x_n \uparrow x_\infty$ a.s., $\rho(-x_n) \uparrow \rho(-x_\infty)$. Denote by \mathcal{R} the set of all convex risk measures with $\rho(0) = 0$ and the Fatou property.

Theorem 2.2 (Biagini and Frittelli [4]) For $\rho \in \mathcal{R}$, we have for $x \in L$,

$$\rho(x) = \sup_{Q \in \mathcal{P}} \{\mathbb{E}_Q[-x] - \rho^*(Q)\}. \quad (2.1)$$

A convex risk measure ρ is said to be **finite** if $\rho(x) < \infty$ for all $x \in L$.

Remark 2.3 In the case of $L = M^\Psi$, it is known that L^+ coincides with the dual of L and the supremum in (2.1) is attained. Moreover, every finite convex risk measure has the Fatou property. See [4] for the detail. The finiteness condition cannot be dropped as we see in Example 2.7 below. If Ψ satisfies the Δ_2 condition: there exist $t_0 > 0$ and $K > 0$ such that $\Psi(2t) \leq K\Psi(t)$ for any $t \geq t_0$, then we have $L^\Psi = M^\Psi$. For $p \in [1, \infty)$, L^p is an example of such cases. \square

A convex risk measure ρ is said to have **the Lebesgue property** if for any sequence $\{x_n\} \subset L$ with $\sup_n \|x_n\|_\infty < \infty$ and $x_n \rightarrow x_\infty$ a.s., it holds that $\rho(x_n) \rightarrow \rho(x_\infty)$ as $n \rightarrow \infty$. Here $\|\cdot\|_\infty$ refers to the L^∞ norm. This definition was introduced in [18] for the $L = L^\infty$ case. Since any continuous linear functional on L can be decomposed into the sum of an element of $L^+ \subset L^1$ and a purely finitely additive signed measure (see [23]), the same argument as the proof of Theorem 2.4 in [18] can apply to have the following result with the aid of Theorem 2.1 above.

Theorem 2.4 For a finite convex risk measure ρ , the following are equivalent:

1. ρ has the Lebesgue property.
2. for any $\alpha > 0$ and a sequence of measurable sets A_n with $P(A_n) \rightarrow 0$, it holds that $\rho(-\alpha 1_{A_n}) \rightarrow 0$ as $n \rightarrow \infty$.
3. for any $c > 0$, the set $\{g \in L_+^*; \rho^*(g) \leq c\}$ is a uniformly integrable subset of L^+ and for any $x \in L$, it holds that

$$\rho(-x) = \max_{Q \in \mathcal{P}} \{\mathbb{E}_Q[x] - \rho^*(Q)\}. \quad (2.2)$$

Note that the Fatou property follows from the Lebesgue property by (2.2).

A convex risk measure is said to be **relevant** if $\rho(-z) > 0$ for any $z \in L_+ \setminus \{0\}$. The relevance was introduced in [12] as a condition for coherent risk measures with the Fatou property to be represented as (2.1) with a set of equivalent probability measures instead of \mathcal{P} .

2.3 Superhedging cost

Here we discuss superhedging cost. Define a functional ρ^0 on L as

$$\rho^0(x) := \inf\{c \in \mathbb{R} \mid \text{there exists } m \in M \text{ such that } c + m + x \geq 0\}. \quad (2.3)$$

Since $\rho^0(-x)$ represents the superhedging cost for a claim x , it gives the upper no-arbitrage pricing bound for x . In fact if a seller could sell x with a price greater than $\rho^0(-x)$, then she could enjoy an arbitrage opportunity by taking a suitable strategy from M . By the same reasoning the lower no-arbitrage pricing bound for x is given by $-\rho^0(x)$.

Lemma 2.5 *The superhedging cost ρ^0 is $(-\infty, \infty]$ -valued if and only if $\bar{L}^* \neq \emptyset$. If ρ^0 is $(-\infty, \infty]$ -valued, then it is a coherent risk measure with*

$$(\rho^0)^*(g) = \begin{cases} 0 & \text{if } g \in \bar{L}^*, \\ \infty & \text{otherwise.} \end{cases}$$

Proof. Suppose that $\bar{L}^* \neq \emptyset$. If there exists $x \in L$ with $\rho^0(x) = -\infty$, then (2.3) implies that for any $c > 0$, we can find $m^c \in M$ such that $-c + m^c + x \geq 0$. This gives $g(x) \geq c$, so that $g(x) = \infty$ for any $g \in \bar{L}^*$. This is a contradiction, so ρ^0 is $(-\infty, \infty]$ -valued. Next, suppose that $\bar{L}^* = \emptyset$. Then there exists a sequence $\{m_n\} \subset M$ such that $\|m_n - 1\| \rightarrow 0$ as $n \rightarrow \infty$. In fact if the closure M^s of M does not include 1, then the Hahn-Banach theorem implies the existence of a continuous linear functional μ such that $\mu(1) > \sup_{m \in M^s} \mu(m)$. The RHS is 0 since M^s is a cone. That $L_- \subset M^s$ implies $\mu \in L_+^*$. This means $\bar{L}^* \neq \emptyset$, which is a contradiction. Now, taking a subsequence if necessary, we may suppose that $\sum_{n=1}^{\infty} \|m_n - 1\| < \infty$. Then for $x := \sum_{n=1}^{\infty} |m_n - 1| \in L$ and for all $N \in \mathbb{N}$,

$$x \geq \sum_{n=1}^N (1 - m_n) = N - \sum_{n=1}^N m_n,$$

which implies that $\rho^0(x) \leq -N$, and so $\rho^0(x) = -\infty$.

Now we see that ρ^0 is a coherent risk measure and calculate $(\rho^0)^*$. The convexity and positive homogeneity of ρ^0 follow from the assumption that M is a convex cone. The monotonicity and cash-invariance are obvious. The fact that $\rho^0(0) \leq 0$ implies that $(\rho^0)^*(g) \geq 0$ for any $g \in L_+^*$. On the other hand, for any $\varepsilon > 0$ and $x \in L$, we can find $m^\varepsilon \in M$ so that $\rho^0(x) + \varepsilon + m^\varepsilon + x \geq 0$. Since $g(m^\varepsilon) \leq 0$ for $g \in \bar{L}^*$, we have $\rho^0(x) + \varepsilon \geq g(-x)$, which implies that

$$\sup_{x \in L} \{g(-x) - \rho^0(x)\} \leq 0.$$

We therefore have $(\rho^0)^*(g) = 0$ for $g \in \bar{L}^*$. For $g \in L_+^* \setminus \bar{L}^*$, there exists $m \in M$ such that $g(m) > 0$. Since M is a cone,

$$(\rho^0)^*(g) = \sup_{x \in L} \{g(-x) - \rho^0(x)\} \geq \sup_{m \in M} \{g(m) - \rho^0(-m)\} \geq \sup_{m \in M} g(m) = \infty.$$

□

For later use, we define for $x \in L$,

$$\widehat{\rho^0}(x) := \begin{cases} \sup_{Q \in \mathcal{Q}} \mathbb{E}_Q[-x] & \text{if } \mathcal{Q} \neq \emptyset \\ -\infty & \text{otherwise.} \end{cases}$$

By definition $\widehat{\rho^0}$ is a coherent risk measure on L belonging to \mathcal{R} if $\mathcal{Q} \neq \emptyset$.

Lemma 2.6 *If $\mathcal{Q} \neq \emptyset$, then $-\rho^0(x) \leq -\widehat{\rho^0}(x) \leq \widehat{\rho^0}(-x) \leq \rho^0(-x)$ for any $x \in L$. Moreover if $\mathcal{Q}^e \neq \emptyset$, then $\widehat{\rho^0}$ is relevant.*

Proof. For any $x \in L$ and $\varepsilon > 0$, there exists $m \in M$ such that $\rho^0(x) + \varepsilon + m + x \geq 0$. Then we have $\mathbb{E}_Q[-x] \leq \rho^0(x) + \varepsilon$ for any $Q \in \mathcal{Q}$. Since $Q \in \mathcal{Q}$ and $\varepsilon > 0$ are arbitrary, we have $\widehat{\rho^0}(x) \leq \rho^0(x)$. It suffices then to observe that $\widehat{\rho^0}(x) + \widehat{\rho^0}(-x) \geq 2\rho^0(0) = 0$ by the convexity.

The relevance under $\mathcal{Q}^e \neq \emptyset$ is shown by noting that

$$\widehat{\rho^0}(-x) = \sup_{Q \in \mathcal{Q}} \mathbb{E}_Q[x] = \sup_{Q \in \mathcal{Q}^e} \mathbb{E}_Q[x].$$

In fact if there exists $Q_1 \in \mathcal{Q}$ with $\mathbb{E}_{Q_1}[x] > \sup_{Q \in \mathcal{Q}^e} \mathbb{E}_Q[x]$, then we have a contradiction since for any $Q_0 \in \mathcal{Q}^e$, $\lambda Q_0 + (1 - \lambda)Q_1 \in \mathcal{Q}^e$ converges to Q_1 in $\sigma(L^+, L)$ as $\lambda \downarrow 0$. □

The following example shows that ρ^0 does not necessarily coincides with $\widehat{\rho^0}$, so is not always represented as (2.1) even though \mathcal{Q} is not empty.

Example 2.7 Let $L = L^p$ with $p \in [1, \infty)$ and take the following set as M :

$$M = \{-z + \mathbb{E}_{Q^0}[z] | z \in L_+\} - L_+,$$

where $Q^0 \in \mathcal{P}$ is arbitrarily fixed. Any element of M is bounded from above. Therefore by the definition of ρ^0 , we have $\rho^0(-z) = \infty$ for $z \in L_+$ which is not bounded from above. It is clear that $Q^0 \in \mathcal{Q}$, so that $\overline{L}^* \neq \emptyset$. Therefore ρ^0 is a coherent risk measure by Lemma 2.5. Moreover $\mathcal{Q} = \{Q^0\}$ since for any $Q \in \mathcal{Q}$, we have $\mathbb{E}_{Q^0}[z] \leq \mathbb{E}_Q[z]$ for any $z \in L_+$, which implies that $Q = Q^0$. Therefore ρ^0 cannot be represented as (2.1).

In fact we can prove that ρ^0 does not have the Fatou property. Let $z \in L_+$ be unbounded from above. Consider the increasing sequence $z_n = z \wedge n$, $n \in \mathbb{N}$. Since $n - z_n \in L_+$, we have $z_n - \mathbb{E}_{Q^0}[z_n] \in M$. It follows that $\rho^0(-z_n) \leq \mathbb{E}_{Q^0}[z_n] \rightarrow \mathbb{E}_{Q^0}[z] < \infty$, while $\rho^0(-z) = \infty$. \square

2.4 Risk indifference prices

Here we recall risk indifference price. Given a convex risk measure ρ , define a functional $I(\rho)$ on L as

$$\begin{aligned} I(\rho)(x) &:= \inf \{c \in \mathbb{R} | \inf_{m \in M} \rho(c + m + x) \leq \inf_{m \in M} \rho(m)\} \\ &= \inf \{c \in \mathbb{R} | \inf_{m \in M} \rho(m + x) - c \leq \inf_{m \in M} \rho(m)\}. \end{aligned} \quad (2.4)$$

Then $I(\rho)(-x)$ describes the risk indifference seller's price for x induced by ρ as introduced in [26]. The idea is explained as follows. If a trader sells a claim x with a price $c > I(\rho)(-x)$, then she can find $\widehat{m} \in M$ such that $\rho(c + \widehat{m} - x) \leq \inf_{m \in M} \rho(m)$. This means that selling the claim with the price does not increase the risk measured by ρ . The following lemma gives a representation of $I(\rho)$. Denote $\check{\rho} := \rho - \inf_{m \in M} \rho(m)$.

Lemma 2.8 *Let ρ be a convex risk measure. If $I(\rho)$ is $(-\infty, \infty]$ -valued, then we have $\inf_{m \in M} \rho(m) \in \mathbb{R}$ and that $I(\rho)$ is a convex risk measure with*

$$I(\rho)^*(g) = \begin{cases} \check{\rho}^*(g) = \rho^*(g) + \inf_{m \in M} \rho(m), & \text{if } g \in \overline{L}^* \\ \infty & \text{otherwise.} \end{cases}$$

If $I(\rho) \in \mathcal{R}$ in addition, then $\mathcal{Q} \neq \emptyset$ and

$$I(\rho)(x) = \sup_{Q \in \mathcal{Q}} \{\mathbb{E}_Q[-x] - \check{\rho}^*(Q)\}. \quad (2.5)$$

Proof. Since $\rho(0) < \infty$ and $0 \in M$, we have $I(\rho)(0) = 0$ or $-\infty$ depending on whether $\inf_{m \in M} \rho(m)$ is finite or $-\infty$. Therefore if $I(\rho) > -\infty$ then $\inf_{m \in M} \rho(m)$ is finite and $I(\rho)(x) = \inf_{m \in M} \rho(x + m) - \inf_{m \in M} \rho(m) = \inf_{m \in M} \check{\rho}(x + m)$. From this the cash-invariance and monotonicity of $I(\rho)$ are obvious. The convexity follows from that M is convex. Since M is a cone, we have

$$\begin{aligned} I(\rho)^*(g) &= \sup_{x \in L} \{g(-x) - I(\rho)(x)\} \\ &= \sup_{m \in M} \sup_{x \in L} \{g(-x) - \check{\rho}(x + m)\} \\ &= \sup_{m \in M} \{g(m) + \check{\rho}^*(g)\} \\ &= \begin{cases} \check{\rho}^*(g) & \text{if } g \in \bar{L}^* \\ \infty & \text{otherwise.} \end{cases} \end{aligned}$$

By Theorem 2.2, we have (2.5) if $I(\rho) \in \mathcal{R}$ and in particular, $Q \neq \emptyset$. \square

3 Good deal valuations

In this section we discuss conditions under which a convex risk measure yields a good deal bound. A good deal bound should be a subinterval of the no-arbitrage pricing bound. We therefore introduce the following definition.

Definition 3.1 A convex risk measure $\rho \in \mathcal{R}$ is said to be a good deal valuation (GDV) if

$$\rho(-x) \in [-\rho^0(x), \rho^0(-x)] \text{ for any } x \in L. \quad (3.1)$$

As mentioned in Introduction, the above definition is given from seller's viewpoint. Nevertheless, (3.1) is equivalent to

$$-\rho(x) \in [-\rho^0(x), \rho^0(-x)] \text{ for any } x \in L, \quad (3.2)$$

which is from buyer's viewpoint. In addition, $-\rho(x) \leq \rho(-x)$ for any $x \in L$ because $\rho(x) + \rho(-x) \geq 2\rho(0) = 0$ by the convexity. For a GDV ρ , a good deal bound may be constructed as $[-\rho(x), \rho(-x)]$, which is a subinterval of $[-\rho^0(x), \rho^0(-x)]$. Note that the upper and lower bounds of a good deal bound may be described by different GDVs.

3.1 Existence of good deal valuations

Here we present a set of equivalent conditions for the existence of a GDV. Denote by \overline{M} the closure of M in $\sigma(L, L^+)$.

Theorem 3.2 *The following are equivalent:*

1. $Q \neq \emptyset$.
2. *There exists a GDV.*
3. $\mathbb{P}(m > 0) < 1$ for any $m \in \overline{M}$.
4. $1 \notin \overline{M}$.

Proof. $1 \Rightarrow 2$: This is from Lemma 2.6.

$2 \Rightarrow 1$: Let ρ be a GDV. Since $\rho(-m) \leq \rho^0(-m) \leq 0$ for any $m \in M$,

$$\rho^*(Q) = \sup_{x \in L} \{\mathbb{E}_Q[-x] - \rho(x)\} \geq \sup_{m \in M} \{\mathbb{E}_Q[m] - \rho(-m)\} \geq \sup_{m \in M} \mathbb{E}_Q[m].$$

Then the cone property of M implies that $\rho^*(Q) = +\infty$ for any $Q \in \mathcal{P} \setminus Q$. If Q is empty, then ρ equals to $-\infty$ identically by (2.1), which contradicts $\rho \in \mathcal{R}$.

$1 \Rightarrow 3$: If there exists $m \in \overline{M}$ such that $P(m > 0) = 1$, then we have $\mathbb{E}_Q[m] > 0$ for any $Q \in \mathcal{P}$, and so $Q = \emptyset$.

$3 \Rightarrow 4$: This holds true clearly.

$4 \Rightarrow 1$: Since $1 \notin \overline{M}$, the Hahn-Banach theorem implies that there exists $z \in L^+$ such that

$$\sup_{m \in \overline{M}} \mathbb{E}[zm] < \mathbb{E}[z]. \quad (3.3)$$

We have $\sup_{m \in \overline{M}} \mathbb{E}[zm] = 0$ because $0 \in M$ and \overline{M} is a cone. Since $L_- \subset \overline{M}$, we have then that $z \in L_+^* \cap L^+$, so that $z/\mathbb{E}[z] \in Q$. \square

Condition 3 in the above theorem is weaker than the no-arbitrage condition. This means that a GDV may exist even if there is an arbitrage opportunity. The following example shows that we cannot replace \overline{M} with M in Conditions 3 and 4.

Example 3.3 We take the Lebesgue measure space on $(0, 1]$ as the underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let u be the random variable given by $u(\omega) := \omega$, and M be given by $\{cu | c \geq 0\} - L_+$. We can see several interesting facts on this example as follows:

1. We consider the following two conditions:

- (a) $\mathbb{P}(m > 0) < 1$ for any $m \in M$,
- (b) $1 \notin M$.

This example satisfies (b), but does not satisfy (a). Replacing M by \overline{M} , the two conditions become equivalent by Theorem 3.2.

- 2. Since $1 \notin M$, we have $\rho^0(0) = 0$. Therefore if we take $L = L^\infty$, then ρ^0 is a finite coherent risk measure. In fact for any $x \in L^\infty$, $-||x||_\infty = \rho^0(||x||_\infty) \leq \rho^0(x) \leq \rho^0(-||x||_\infty) = ||x||_\infty$ by monotonicity. On the other hand, ρ^0 is not a convex risk measure on $L = L^p$ with $p \in [1, \infty)$ since $\overline{L}^* = Q$ is empty. Note that for $x(\omega) := \log \omega$, we have $\rho^0(-x) = -\infty$.
- 3. Notice that Q is empty despite that the above Condition (b) holds. We therefore need to take the closure of M in Condition 4 of Theorem 3.2. In fact, considering the sequence $m_n := (nu) \wedge 1$, m_n converges to 1, and so this example does not satisfy Conditions 3 nor 4.

□

3.2 Equivalent conditions for good deal valuations

Here we present conditions for a given ρ to be a GDV. The main contribution of the following theorem, is to show the equivalence between GDVs and risk indifference prices.

Theorem 3.4 *For any $\rho \in \mathcal{R}$, the following conditions are equivalent:*

- 1. ρ is a GDV.
- 2. $\rho(-m) \leq 0$ for any $m \in M$.
- 3. There exists a function $c : Q \rightarrow \mathbb{R}$ such that for any $x \in L$,

$$\rho(x) = \sup_{Q \in Q} \{\mathbb{E}_Q[-x] - c(Q)\}.$$

- 4. There exists $\eta \in \mathcal{R}$ such that $\rho = I(\eta)$.
- 4'. $\rho = I(\rho)$, that is, ρ is a fixed point of I .
- 5. $\rho(-x) \in [-\widehat{\rho^0}(x), \widehat{\rho^0}(-x)]$ for any $x \in L$.

6. $\{\rho^0 \leq 0\} \subset \{\rho \leq 0\}$.
7. $Q \supset \{Q \in \mathcal{P} | \rho^*(Q) < +\infty\}$.
8. *There exists a convex set $A \subset L$ including 0 with $A+L_+ \subset A$ and $A \cap \mathbb{R} = \mathbb{R}_+$ such that for any $x \in L$,*

$$\rho(x) = \inf\{c \in \mathbb{R} | \text{there exists } m \in M \text{ such that } c + m + x \in A\}. \quad (3.4)$$

Proof. 1 \Rightarrow 2: This is because $\rho(-m) \leq \rho^0(-m) \leq 0$ for any $m \in M$ by the definitions of GDV and ρ^0 .

2 \Rightarrow 7: We have

$$\rho^*(Q) = \sup_{x \in L} \{\mathbb{E}_Q[-x] - \rho(x)\} \geq \sup_{m \in M} \{\mathbb{E}_Q[m] - \rho(-m)\} \geq \sup_{m \in M} \mathbb{E}_Q[m].$$

Since M is a cone, we have $\rho^*(Q) = \infty$ for any $Q \in \mathcal{P} \setminus Q$.

7 \Rightarrow 3: This is from Theorem 2.2.

3 \Rightarrow 4' and 4: Since $\rho \in \mathcal{R}$, we have

$$\rho(-m) = \sup_{Q \in Q} \{\mathbb{E}_Q[m] - c(Q)\} \leq - \inf_{Q \in Q} c(Q) = \rho(0) = 0$$

for any $m \in M$. Then, by the convexity, we have $\rho(m) + \rho(-m) \geq 2\rho(0) = 0$ and so, $\inf_{m \in M} \rho(m) = 0$. Therefore,

$$I(\rho)(x) = \inf_{m \in M} \rho(m+x) - \inf_{m \in M} \rho(m) \leq \rho(x) \quad (3.5)$$

and

$$I(\rho)(x) = \inf_{m \in M} \sup_{Q \in Q} \{\mathbb{E}_Q[-m-x] - c(Q)\} \geq \sup_{Q \in Q} \{\mathbb{E}_Q[-x] - c(Q)\} = \rho(x).$$

4 \Rightarrow 5: By Lemma 2.8, $\rho = I(\eta)$ is represented as

$$\rho(x) = \sup_{Q \in Q} \{\mathbb{E}_Q[-x] - \check{\eta}^*(Q)\}.$$

Since $\rho(0) = 0$, we have $\check{\eta}^*(Q) \geq 0$. Therefore,

$$\widehat{\rho^0}(-x) = \sup_{Q \in Q} \mathbb{E}_Q[x] \geq \sup_{Q \in Q} \{\mathbb{E}_Q[x] - \check{\eta}^*(Q)\} = \rho(-x)$$

for all $x \in L$. It suffices then to recall that $\rho(x) + \rho(-x) \geq 2\rho(0) = 0$ by the convexity.

5 \Rightarrow 1: This is from Lemma 2.6.

3 \Rightarrow 6: For any $x \in \{\rho^0 \leq 0\}$, Lemma 2.6 implies that $\sup_{Q \in \mathcal{Q}} \mathbb{E}_Q[-x] = \widehat{\rho^0}(x) \leq 0$. We have then

$$\rho(x) = \sup_{Q \in \mathcal{Q}} \{\mathbb{E}_Q[-x] - c(Q)\} \leq \sup_{Q \in \mathcal{Q}} \{-c(Q)\} = \rho(0) = 0.$$

6 \Rightarrow 2: This is because $\rho^0(-m) \leq 0$ by definition.

4' \Rightarrow 8: Taking $A = \{\rho \leq 0\}$ and noting that $\inf_{m \in M} \rho(m) = 0$, we have

$$\begin{aligned} \rho(x) &= I(\rho)(x) = \inf_{m \in M} \rho(m+x) = \inf\{c \in \mathbf{R} \mid \inf_{m \in M} \rho(m+x) \leq c\} \\ &\leq \inf\{c \in \mathbf{R} \mid \text{there exists } m \in M \text{ such that } \rho(m+x) \leq c\} \\ &= \inf\{c \in \mathbf{R} \mid \text{there exists } m \in M \text{ such that } c+m+x \in A\} \\ &\leq \inf\{c \in \mathbf{R} \mid c+x \in A\} = \rho(x). \end{aligned}$$

8 \Rightarrow 2: This is obvious. □

Remark 3.5 Denote by ρ_A the RHS of (3.4). In [17] and [25], the set A is given as an acceptance set and ρ_A is considered as a functional describing a good deal bound. Therefore they appear to treat a special class of convex risk measures but Theorem 3.4 shows that it is the only class giving good deal bounds. The representation of GDV as ρ_A is important in that it implies robustness of GDV to quantitative specification of investor's risk preference. Notice however that ρ_A is not necessarily normalized. As long as treating ρ_A , the condition defining GDV is equivalent to the no-cashout condition (NC) introduced in [25]: $\rho_A(-x) \geq -\rho^0(x)$ for any $x \in L$. In fact for any $x \in L$,

$$\begin{aligned} \rho^0(x) &= \inf\{c \in \mathbf{R} \mid \text{there exists } m \in M \text{ such that } c+m+x \in L_+\} \\ &\geq \inf\{c \in \mathbf{R} \mid \text{there exists } m \in M \text{ such that } c+m+x \in A\} \\ &= \rho_A(x), \end{aligned}$$

that is, the upper estimate for $\rho_A(-x)$ holds automatically. The convexity of ρ_A implies that NC is equivalent to $\rho_A(0) = 0$. Theorem 6.1(0th FTAP) of [25] states, in a more abstract setting, a condition under which $\rho_A(0) = 0$. □

As mentioned in Introduction, many papers ([14], [20], [23], [26],...) treated risk indifference prices and some of them showed that a risk indifference price yields a good deal bound. On the other hand, Theorem 3.4 showed that a GDV is always a risk indifference price. It therefore supports the use of the

operator I in constructing a good deal bound. We utilized however that a GDV has the Fatou property by definition. It should be noted that $I(\rho)$ does not necessarily have the Fatou property even if $\rho \in \mathcal{R}$. In other words, the operation does not necessarily preserve the Fatou property (see Example 3.8 below). Now we remark that it preserves the Lebesgue property that also could be regarded as a natural continuity requirement for good deal bounds as well as the Fatou property.

Proposition 3.6 *Let ρ be a finite convex risk measure with the Lebesgue property and suppose that there exists $Q^0 \in \mathcal{Q}$ such that $\rho^*(Q^0) < \infty$. Then, $I(\rho)$ is a finite GDV with the Lebesgue property.*

Proof. By Theorem 2.4 and the existence of $Q^0 \in \mathcal{Q}$ such that $\rho^*(Q^0) < \infty$, we have, for any $x \in L$ and $m \in M$,

$$\begin{aligned} \rho(x + m) &= \max_{Q \in \mathcal{P}} \{E_Q[-x - m] - \rho^*(Q)\} \geq E_{Q^0}[-x - m] - \rho^*(Q^0) \\ &\geq E_{Q^0}[-x] - \rho^*(Q^0) > -\infty. \end{aligned}$$

Therefore $I(\rho)$ is $(-\infty, \infty]$ -valued by (2.4), and so it is a convex risk measure by Lemma 2.8. Since ρ is finite, so is $I(\rho)$ by (2.4). Moreover for any $m \in M$, we have

$$I(\rho)(-m) = \inf_{m' \in M} \rho(-m + m') - \inf_{m' \in M} \rho(m') \leq \inf_{m' \in M} \rho(m') - \inf_{m' \in M} \rho(m') = 0. \quad (3.6)$$

Therefore by Theorem 3.4, it only remains to show that $I(\rho)$ has the Fatou property. By (2.2), it suffices to see that $I(\rho)$ has the Lebesgue property. Note that $I(\rho)(m) \geq 0$ for any $m \in M$ by the convexity. For any $\alpha > 0$, $\epsilon > 0$ and a sequence of measurable sets A_n with $P(A_n) \rightarrow 0$, we have that

$$\begin{aligned} 0 \leq I(\rho)(-\alpha 1_{A_n}) &= \inf_{m \in M} \rho(m - \alpha 1_{A_n}) - \inf_{m \in M} \rho(m) \\ &\leq (1 - \epsilon) \inf_{m \in M} \rho\left(\frac{m}{1 - \epsilon}\right) + \epsilon \rho\left(-\frac{\alpha}{\epsilon} 1_{A_n}\right) - \inf_{m \in M} \rho(m) \\ &\rightarrow -\epsilon \inf_{m \in M} \rho(m) \end{aligned} \quad (3.7)$$

as $n \rightarrow \infty$ by the Lebesgue property of ρ . Since ϵ is arbitrary, we conclude the Lebesgue property of $I(\rho)$ by Theorem 2.4. \square

Proposition 3.7 *For a finite convex risk measure ρ , the following are equivalent:*

1. ρ is a GDV with the Lebesgue property.

2. there exists a convex risk measure η with the Lebesgue property, $\rho = I(\eta)$.

Proof. $1 \Rightarrow 2$: This is because $\rho = I(\rho)$ by Theorem 3.4.

$2 \Rightarrow 1$: By Lemma 2.8, we have $\inf_{m \in M} \eta(m) \in \mathbb{R}$, and so

$$I(\eta)(x) = \inf_{m \in M} \eta(x + m) - \inf_{m \in M} \eta(m).$$

In particular we have (3.6) and (3.7) with η instead of ρ . By the finiteness of $\rho = I(\eta)$, Theorem 2.4 can be applied to have the result. \square

Example 3.8 Consider $L = L^\infty(\mathbb{R}, \mathcal{F}, \mathbb{P})$, where \mathbb{P} is a normal distribution on \mathbb{R} . Let $Q \in \mathcal{P}$ have a compact support and define a sequence $\{Q_n\} \subset \mathcal{P}$ by $Q_n(A) := Q(A - n)$ for $A \in \mathcal{F}$, $n \in \mathbb{N}$. Since $\{g \in L_+^* | g(1) = 1\}$ is weak-* compact, there exists a cluster point μ of $\{Q_n\}$. Since $\{Q_n\}$ is not tight, $\mu \notin \mathcal{P}$. Consider $M = \{x \in L | \mu(x) \leq 0\}$. Observe that $\bar{L}^* = \{\mu\}$. In fact if there exists $v \in \bar{L}^*$ and $x \in L$ with $v(x) > \mu(x)$, then $y := x - \mu(x) \in M$ and $v(y) > 0$, which is a contradiction. Now consider $\rho \in \mathcal{R}$ defined as $\rho(-x) = \sup_{Q \in \mathcal{P}} \mathbb{E}_Q[x]$. Let us show that $\rho^*(\mu) = 0$. By $\rho(0) = 0$ we have $\rho^*(\mu) \geq 0$ and $\rho^*(Q_n) = 0$. If $\rho(\mu) > 0$, then there exists $x \in L$ such that $\mu(x) > \sup_{Q \in \mathcal{P}} \mathbb{E}_Q[x]$, which contradicts that μ is a cluster point of Q_n . By the same reason, we have also that for any $m \in M$ and $x \in L$, $\rho(m + x) \geq \mu(-m - x) \geq -\mu(x)$, so that $I(\rho)$ is finite. By Lemma 2.8, $I(\rho)^*(g) = \infty$ for any $g \in L_+^* \setminus \bar{L}^*$, so by Theorem 2.1, we have $I(\rho)(-x) = \mu(x)$ for all $x \in L$. To see that $I(\rho)$ does not have the Fatou property, consider the increasing sequence $x_n := 1_{(-\infty, n]}$. Then $I(\rho)(-x_n) = 0$ while $I(\rho)(-x_\infty) = 1$. \square

3.3 Shortfall risk measures

Here we treat shortfall risk measure as an application. We presume an investor who sells a claim x . When she sells x with price c and selects $m \in M$ as her strategy, her final cash-flow is $c + m - x$, and so its shortfall is $(c + m - x) \wedge 0$. In general, shortfall risk is defined as a weighted expectation of the shortfall with a loss function. A loss function is a continuous strictly increasing convex function $l : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ with $l(0) = 0$. This represents the seller's attitude towards risk. To suppress the shortfall risk less than a certain level $\delta > 0$ which she can endure, the least price she can accept is given as

$$\rho_l(-x) := \inf\{c \in \mathbb{R} | \text{there exists } m \in M \text{ such that } E[l((c + m - x)^-)] \leq \delta\}. \quad (3.8)$$

As shown in [1] and [16], ρ_l is a convex risk measure and it has the Fatou property under mild conditions. However, it is not a GDV as $\rho_l(0) \neq 0$:

Proposition 3.9 *Any shortfall risk measure is not a GDV.*

Proof. For any shortfall risk measure ρ_l , (3.8) implies that

$$\begin{aligned}\rho_l(0) &= \inf\{c \in \mathbb{R} \mid \text{there exists } m \in M \text{ such that } E[l((c+m)^-)] \leq \delta\} \\ &\leq \inf\{c \in \mathbb{R} \mid l(c^-) \leq \delta\} = -l^{-1}(\delta) < 0.\end{aligned}$$

Hence, $\rho_l \notin \mathcal{R}$, from which ρ_l is not a GDV. \square

Now we show that a normalized shortfall risk measure can be a GDV. Define $\widehat{\rho}_l$ as $\widehat{\rho}_l(x) := \rho_l(x) - \rho_l(0)$.

Proposition 3.10 *If $\widehat{\rho}_l \in \mathcal{R}$, then $\widehat{\rho}_l$ is a GDV.*

Proof. In light of Theorem 3.4, it suffices to see $I(\widehat{\rho}_l) = \widehat{\rho}_l$. Since $\widehat{\rho}_l(m) \geq -\widehat{\rho}_l(-m) \geq 0$ for $m \in M$, we have $\inf_{m \in M} \widehat{\rho}_l(m) = 0$, and so $I(\widehat{\rho}_l)(x) = \inf_{m \in M} \rho_l(m+x) - \rho_l(0)$. Now let us observe that $\inf_{m \in M} \rho_l(m+x) = \rho_l(x)$ for any $x \in L$. $\inf_{m \in M} \rho_l(m+x) \leq \rho_l(x)$ holds clearly. Fix $m \in M$ and $c > \rho_l(m+x)$ arbitrarily. Then there exists $m' \in M$ such that $E[l((c+m'+m+x)^-)] \leq \delta$. Since $m' + m \in M$, we have $c \geq \rho_l(x)$. \square

4 Relevant good deal valuations

4.1 Fundamental Theorem of Asset Pricing

We have seen that the condition $\mathcal{Q} \neq \emptyset$ is equivalent to the existence of a GDV. Example 4.1 below shows that $\mathcal{Q} \neq \emptyset$ is not sufficient to rule out arbitrage opportunities in general.

Example 4.1 Let $A \in \mathcal{F}$ with $P(A) \in (0, 1)$, $m' := 1_A$ and $M = \{cm' \mid c \geq 0\} - L_+$. Any probability measure $Q \in \mathcal{P}$ with $Q(A) = 0$ is in \mathcal{Q} . On the other hand, cm' with $c > 0$ brings an arbitrage opportunity. \square

Kreps [21] showed that $\mathcal{Q}^e \neq \emptyset$ is equivalent to NFL, that is, $\overline{M} \cap L_+ = \{0\}$. Here we prove that $\mathcal{Q}^e \neq \emptyset$ is equivalent to the existence of a relevant GDV, that is, a relevant convex risk measure which is a GDV.

Theorem 4.2 (FTAP) *The following are equivalent:*

1. $Q^e \neq \emptyset$.
2. $\overline{M} \cap L_+ = \{0\}$.
3. *There exists a relevant GDV.*

Proof. $2 \Rightarrow 1$: For any $a, b \in \mathbb{R}$, the set $\{x \in L | a \leq x \leq b\}$ is compact in $\sigma(L, L^\dagger)$. In fact if $L = L^\infty$, then $L^\dagger = L^1$ and $\sigma(L, L^\dagger)$ is the weak-* topology. The compactness then follows from the Banach-Alaoglu theorem. It suffices then to notice that $L^\infty \subset L$, $L^\dagger \subset L^1$ as sets of random variables and the natural inclusion $(L^\infty, \sigma(L^\infty, L^1)) \rightarrow (L, \sigma(L, L^\dagger))$ is continuous. Therefore we can prove the existence of an element of Q^e in exactly the same manner as in the proof of Theorem 5.2.3 of [13].

$1 \Rightarrow 3$: This is from Lemma 2.6.

$3 \Rightarrow 2$: Let ρ be a relevant GDV. We have $\rho(x) = \sup_{Q \in Q} \{\mathbb{E}_Q[-x] - c(Q)\}$ by Item 3 of Theorem 3.4. Since $\rho(-z) > 0$ for all $z \in L_+$ by the relevance, it suffices to see that $\rho(-\overline{m}) \leq 0$ for any $\overline{m} \in \overline{M}$. If there exists $\overline{m} \in \overline{M}$ with $\rho(-\overline{m}) > 0$, there exists $Q \in Q$ such that $\mathbb{E}_Q[\overline{m}] > c(Q) \geq \sup_{m \in M} \mathbb{E}_Q[m]$. The last inequality is from the fact that $\rho(-m) \leq 0$ for all $m \in M$. This contradicts that \overline{m} is in the closure of M in $\sigma(L, L^\dagger)$. \square

Now we give a set of equivalent conditions for GDV to be relevant. Let

$$\begin{aligned} M^\rho &:= \{x - \rho(-x) | x \in L, \rho(-x) < \infty\} - L_+ = \{x \in L | \rho(-x) = 0\} - L_+ \\ &= \{x \in L | \rho(-x) \leq 0\}. \end{aligned} \tag{4.1}$$

Note that M^ρ is a convex set including M and interpreted as the set of the 0-attainable claims of an extended market where an investor offers prices for all $x \in L$ by using ρ as her pricing functional. In light of Theorem 2.2, M^ρ is closed in $\sigma(L, L^\dagger)$. Therefore NFL for this extended market is $M^\rho \cap L_+ = \{0\}$.

Theorem 4.3 *For a GDV ρ , the following are equivalent:*

1. ρ is relevant.
2. $-\widehat{\rho^0}(x - z) < \rho(-x)$ for any $x \in L$ and $z \in L_+ \setminus \{0\}$.
3. $-\rho^0(x - z) < \rho(-x)$ for any $x \in L$ and $z \in L_+ \setminus \{0\}$.
4. $M^\rho \cap L_+ = \{0\}$.

Proof. 1 \Rightarrow 2: By the relevance and Theorem 3.4, for any $z \in L_+ \setminus \{0\}$, there exists $Q(z) \in \mathcal{Q}$ such that $\mathbb{E}_{Q(z)}[z] > \rho^*(Q(z))$. Therefore,

$$-\widehat{\rho^0}(x - z) = \inf_{Q \in \mathcal{Q}} \mathbb{E}[x - z] \leq \mathbb{E}_{Q(z)}[x - z] < \mathbb{E}_{Q(z)}[x] - \rho^*(Q(z)) \leq \rho(-x).$$

2 \Rightarrow 3: This is from Lemma 2.6.

3 \Rightarrow 1: For a given $z \in L_+ \setminus \{0\}$, let $x = z$.

1 \Rightarrow 4: This is because ρ separates M^ρ and $L_+ \setminus \{0\}$.

4 \Rightarrow 1: If ρ is not relevant, then there exists $z \in L_+ \setminus \{0\}$ such that $\rho(-z) = 0$. In particular $z \in M^\rho$, which is a contradiction. \square

Remark 4.4 Item 3 of Theorem 4.3 is the no-near-arbitrage condition (NNA) introduced in [25]. Theorem 6.2 of [25] states a condition under which ρ_A satisfies NNA. Proposition 4.5 below may be regarded as its counterpart. \square

Proposition 4.5 *Let ρ be a GDV. If there exists $Q_0 \in \mathcal{Q}^e$ such that $\rho^*(Q_0) = 0$, then ρ is relevant. The reverse implication holds true if ρ is coherent.*

Proof. The relevance is clear from Theorem 2.2. The converse is the Halmos-Savage theorem (see e.g. [12]). \square

Note that for $Q \in \mathcal{P}$ and $\rho \in \mathcal{R}$, $\rho^*(Q) = 0$ is equivalent to that $-\rho(x) \leq \mathbb{E}_Q[x] \leq \rho(-x)$ for all $x \in L$. Therefore such Q is interpreted as a consistent pricing kernel of the extended market M^ρ . The following example shows that the coherence in the second assertion of Proposition 4.5 cannot be dropped. In other words, there is no strictly positive consistent pricing kernel in general even if M^ρ satisfies NFL: $M^\rho \cap L_+ = \{0\}$.

Example 4.6 Set $\Omega = \{\omega_1, \omega_2\}$, and $M = L_-$. Denoting $q := Q(\{\omega_1\})$, we can identify q with $Q \in \mathcal{Q}$. From this viewpoint, \mathcal{Q} and \mathcal{Q}^e are corresponding to $[0, 1]$ and $(0, 1)$ respectively. Consider $\rho(-x) = \sup_{Q \in \mathcal{Q}} \{\mathbb{E}_Q[x] - c(Q)\}$ with $c(Q) = q^2$. Then we have $\rho^*(Q) = c(Q)$. Denoting $z_i := z(\omega_i)$ for $i = 1, 2$, we have $\rho(-z) = \sup_{Q \in \mathcal{Q}} \{\mathbb{E}_Q[z] - c(Q)\} = \sup_{q \in [0, 1]} \{qz_1 + (1 - q)z_2 - q^2\} = \sup_{q \in (0, 1)} \{qz_1 + (1 - q)z_2 - q^2\} > 0$ for any $z \in L_+ \setminus \{0\}$. Thus, ρ is a noncoherent relevant GDV. On the other hand, there is no $q \in (0, 1)$ with $c(Q) = 0$. \square

Remark 4.7 In [5], NFL refers to the condition that

$$\overline{\text{cone}(M^\rho)} \cap L_+ = \{0\},$$

where $\overline{\text{cone}(M^\rho)}$ is the closure of $\text{cone}(M^\rho) = \{\lambda m; m \in M^\rho, \lambda \geq 0\}$ in $\sigma(L, L^+)$, which is a different condition to $M^\rho \cap L_+ = \{0\}$ unless ρ is coherent. This alternative definition of NFL enabled to establish the equivalence between NFL of ρ and the existence of $Q_0 \in \mathcal{Q}^e$ with $\rho^*(Q_0) = 0$ in [5]. In fact since $\overline{\text{cone}(M^\rho)}$ becomes a cone, the same argument as the proof of $2 \Rightarrow 1$ of Theorem 4.2 can apply to have $Q_0 \in \mathcal{Q}^e$ with $\mathbb{E}_{Q_0}[m] \leq 0$ for all $m \in M^\rho$. Since $x - \rho(-x) \in M^\rho$ for all $x \in L$, we have $\rho^*(Q_0) = 0$. Note however that $\overline{\text{cone}(M^\rho)}$ does not have any interpretation as the set of the 0-attainable claims in general. For instance, in the model of the preceding example, we can find $x \in L$ with $\rho(-x) \leq 0$ and $\lambda > 0$ satisfying $\rho(-\lambda x) > 0$. Therefore, it seems not adequate, from economical point of view, to adapt such a definition of NFL. Consequently, the existence of Q_0 with $\rho^*(Q_0) = 0$ may not be considered as a necessary condition for ρ to be a reasonable pricing functional.

4.2 When are all good deal valuations relevant?

As seen in Theorem 4.3, when we extend the underlying market M to M^ρ by using a GDV ρ as pricing functional, the extended market M^ρ remains to satisfy NFL if and only if ρ is relevant. Therefore markets in which any GDV is relevant are stable against such extensions of the market. Here we study necessary and (or) sufficient conditions under which all (coherent) GDVs are relevant.

Theorem 4.8 *Suppose $\mathcal{Q}^e \neq \emptyset$ and consider the following conditions:*

1. *Any GDV is relevant.*
2. *$\widehat{\rho^0}(z) < 0$ for any $z \in L_+ \setminus \{0\}$.*
3. *$\mathcal{Q} = \mathcal{Q}^e$.*
- 3' *$\mathcal{Q} = \mathcal{Q}^e$ and \mathcal{Q}^e is $\sigma(L^+, L)$ -compact.*
4. *Any coherent GDV is relevant.*

Then, we have $1 \Leftrightarrow 2$, $2 \Rightarrow 3$, $3' \Rightarrow 2$, $3 \Leftrightarrow 4$.

Proof. 1 \Rightarrow 2: Assume that there exists $z_0 \in L_+ \setminus \{0\}$ such that $\widehat{\rho}^0(z_0) = 0$. Then $\inf_{Q \in \mathcal{Q}} \mathbb{E}_Q[z_0] = 0$, so that we can define $\rho \in \mathcal{R}$ as

$$\rho(-x) = \sup_{Q \in \mathcal{Q}} \{\mathbb{E}_Q[x] - \mathbb{E}_Q[z_0]\}.$$

This is a GDV by Theorem 3.4 but not relevant. In fact $\rho(-z_0) = 0$.

2 \Rightarrow 1: Let ρ be a GDV. Then by Item 5 of Theorem 3.4, $\rho(-z) \geq -\widehat{\rho}^0(z) > 0$ for any $z \in L_+ \setminus \{0\}$.

2 \Rightarrow 3: If $\mathcal{Q} \neq \mathcal{Q}^e$, then there exists $Q^* \in \mathcal{Q} \setminus \mathcal{Q}^e$. Denoting $A = \{dQ^*/d\mathbb{P} > 0\}$, $\widehat{\rho}^0(1_{A^c}) = \sup_{Q \in \mathcal{Q}} \mathbb{E}_Q[-1_{A^c}] \geq \mathbb{E}_{Q^*}[-1_{A^c}] = 0$, while $1_{A^c} \in L_+ \setminus \{0\}$.

3' \Rightarrow 2: By compactness we have for any $z \in L_+ \setminus \{0\}$,

$$\widehat{\rho}^0(z) = \sup_{Q \in \mathcal{Q}} \mathbb{E}_Q[-z] = \sup_{Q \in \mathcal{Q}^e} \mathbb{E}_Q[-z] = \max_{Q \in \mathcal{Q}^e} \mathbb{E}_Q[-z] < 0.$$

3 \Rightarrow 4: Any coherent GDV ρ is represented as $\rho(x) = \sup_{Q \in \widehat{\mathcal{Q}}} \mathbb{E}_Q[-x]$, for some convex set $\widehat{\mathcal{Q}} \subset \mathcal{Q} = \mathcal{Q}^e$. Therefore ρ is relevant.

4 \Rightarrow 3: If $\mathcal{Q} \neq \mathcal{Q}^e$ then we can take Q^* and A in the same way as “2 \Rightarrow 3”. Let $\rho(x) = \sup_{Q \in \mathcal{Q}, Q(A)=1} \mathbb{E}_Q[-x]$. Then $\rho \in \mathcal{R}$ since $Q^*(A) = 1$. By Theorem 3.4, ρ is a coherent GDV but not relevant since $\rho(1_{A^c}) = 0$. \square

The implications “3 \Rightarrow 3'”, “3 \Rightarrow 1 (or 2)” and “2 \Rightarrow 3'” in Theorem 4.8 do not hold in general. We illustrate counterexamples.

Example 4.9 We give an example satisfying Item 3 of Theorem 4.8 which does not satisfy Items 1 nor 3'. Set $\Omega = \mathbb{R}$, $L = L^\infty$ and $\mathbb{P}(du) = \phi(u)du$, where $\phi(u)$ is the standard normal density. We consider the set of the mixed normal distributions. Let V be the set of all probability measures on $(0, \infty)$,

$$Q_\mu(du) := \int \frac{1}{\sqrt{v}} \phi(u/\sqrt{v}) \mu(dv) du$$

for $\mu \in V$, and $\widehat{\mathcal{Q}} := \{Q_\mu | \mu \in V\}$. Define M as

$$M = \{m \in L^\infty | \mathbb{E}_Q[m] \leq 0 \text{ for any } Q \in \widehat{\mathcal{Q}}\}.$$

Note that all bounded odd functions are in M and $\widehat{\mathcal{Q}} \subset \mathcal{Q}^e \subset \mathcal{Q}$. Now we show that $\widehat{\mathcal{Q}}$ is $\sigma(L^1, L^\infty)$ -closed. Let $\{\mu_n\} \subset V$ be a sequence with $Q_{\mu_n} \rightarrow Q$ in $\sigma(L^1, L^\infty)$. Denote $y_w(u) := e^{i w u}$ for any $w, u \in \mathbb{R}$, where $i = \sqrt{-1}$. We have

$$\mathbb{E}_{Q_{\mu_n}}[y_w] = \int e^{-\frac{w^2}{2}} \mu_n(dv),$$

which has the form of the Laplace transform of μ_n . Since $E_{Q_{\mu_n}}[y_w] \rightarrow \mathbb{E}_Q[y_w]$ and $\lim_{w \rightarrow 0} \mathbb{E}_Q[y_w] = 1$, the continuity theorem of Laplace transforms (see Theorem XIII.1.2 of [15]) implies the existence of $\mu \in V$ such that

$$\mathbb{E}_Q[y_w] = \int e^{-\frac{w}{2}v^2} \mu(dv),$$

which is the characteristic function of an element of \widehat{Q} . Hence, $Q \in \widehat{Q}$.

Note that $\widehat{Q} = Q^e = Q$. In fact if there exists $Q^* \in Q \setminus \widehat{Q}$, by the Hahn-Banach theorem there exists $x \in L$ such that $\mathbb{E}_{Q^*}[x] > \sup_{Q \in \widehat{Q}} \mathbb{E}_Q[x] =: \alpha$. However, $x - \alpha \in M$ and $\mathbb{E}_{Q^*}[x - \alpha] > 0$, which contradicts $Q^* \in Q$. On the other hand, Q is not compact. In fact for the sequence $\mu_n := \delta_{1/n}$ for $n \in \mathbb{N}$, where δ_u is the Delta measure concentrated on $\{u\}$, $\{Q_{\mu_n}\}$ does not have a cluster point in \widehat{Q} .

Finally, we construct a GDV ρ which is not relevant. Letting $y(u) := u^2$, we define ρ as $\rho(-x) = \sup_{Q \in Q} \{\mathbb{E}_Q[x] - c(Q)\}$ with $c(Q) = \mathbb{E}_Q[y]$. Obviously, we have $\rho(0) = 0$ and $\rho(-y) = 0$. \square

Example 4.10 Here we see that the implication “2 \Rightarrow 3’” in Theorem 4.8 does not hold. We modify Example 4.9 as follows. Let $\mu_0 \in V$ be fixed and $\widehat{Q}_0 := \{Q_v | v = (\mu_0 + \mu)/2, \mu \in V\}$. By the same argument as in Example 4.9, we can prove the closedness and noncompactness of \widehat{Q}_0 and that $\widehat{Q}_0 = Q = Q^e$. This model however satisfies Item 2 of Theorem 4.8 since

$$\widehat{\rho}^0(z) = \sup_{Q \in \widehat{Q}_0} \mathbb{E}_Q[-z] = \frac{1}{2} E_{Q_{\mu_0}}[-z] + \frac{1}{2} \sup_{\mu \in V} E_{Q_\mu}[-z] \leq \frac{1}{2} E_{Q_{\mu_0}}[-z] < 0.$$

\square

We conclude the paper with one more example, which is a simple model taking transaction cost into account. In the following example, a model satisfying Item 3’ of Theorem 4.8 is constructed.

Example 4.11 Let $\Omega = \{\omega_0, \omega_1, \dots, \omega_n\}$ and the Arrow-Debreu securities for the n states $\omega_1, \dots, \omega_n$ be tradable in a market subject to bid-ask spread. Denote by $a_{1,j}$, $a_{-1,j}$ the ask and bid prices for the state ω_j respectively for each $j = 1, \dots, n$. Let $D := \{-1, 1\}^n$. If $a_{-1,j} \geq 0$ for each j and $\sum_j a_{1,j} \leq 1$, then for any $d \in D$, a probability measure Q_d on Ω is uniquely determined by $Q_d(\{\omega_j\}) = a_{d(j),j}$ for $j = 1, \dots, n$, and $Q_d(\{\omega_0\}) = 1 - \sum_{j=1}^n a_{d(j),j}$.

Now let

$$M = \{x \in L | \mathbb{E}_d[x] \leq 0 \text{ for all } d \in D\} = \left\{x - \max_{d \in D} \mathbb{E}_d[x] | x \in L\right\} - L_+,$$

where \mathbb{E}_d is the expectation under Q_d . Note that any cash-flow $x \in L$ can be uniquely represented as a sum of a constant and the Arrow Debreu securities and that the price for replicating x is $\max_{d \in D} \mathbb{E}_d[x]$. Therefore M is actually the set of the 0-attainable claims in this market. By the same separation argument as in the preceding examples, we can show

$$Q = \left\{ \sum_{d \in D} \lambda_d Q_d | \lambda_d \geq 0 \text{ for all } d \in D \text{ and } \sum_{d \in D} \lambda_d = 1 \right\}.$$

This set is compact because the set of (λ_d) is a finite dimensional simplex. If $\sum_j a_{1,j} < 1$ in addition, then $Q = Q^e$ and so, Item 3' of Theorem 4.8 is satisfied. Consequently, any GDV in this market is relevant. Remark that $\sum_j a_{1,j} < 1$ is a condition which requires market makers not to offer a set of prices which leads an apparent arbitrage opportunity for themselves. \square

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